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C.R. Rao and D.N. Shanbhag

7. PERFORMING ORGANIZATION NAMES(S) AND ADDRESS(ES)
Center for Multivariate Analysis
417 Classroom Building
Penn State University
University Park, PA 16802

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EXTENSIONS OF A CHARACTERIZATION OF AN EXPONENTIAL DISTRIBUTION BASED ON A CENSORED ORDERED SAMPLE

C.R. Rao and D.N. Shanbhag

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Center for Multivariate Analysis
417 Classroom Building
Penn State University
University Park, PA 16802

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EXTENSIONS OF A CHARACTERIZATION OF AN EXPONENTIAL DISTRIBUTION BASED ON A CENSORED ORDERED SAMPLE

C.Radhakrishna Rao
Department of Statistics
The Pennsylvania State University
University Park, PA 16802

and

D.N. Shanbhag
University of Sheffield
Sheffield, UK

ABSTRACT

Dufour gave a conjecture on a characterization of an arbitrary exponential distribution based on a right-censored ordered sample. The conjecture was shown to be true by Leslie and van Eeden (1993) in the case when the number of censored observations is no larger than $(1/3)n - 1$, and by Xu and Yang (1995) and Rao and Shanbhag (1995) when the number is less than or equal to $\max\{0, n-5\}$, where $n(\geq 3)$ is the sample size. In the present paper we give extended versions of the results concerning the Dufour conjecture.

Key words: Characterizations; Dufour conjecture; Exponential distribution; Integrated Cauchy functional equation; Order statistics.

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1. INTRODUCTION

Let $X_1, ..., X_n$ be $n(\geq 3)$ positive random variables defined on a probability space and let r be an integer satisfying $2 \leq r \leq n$. Also, let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ denote the order statistics relative to $X_1, X_2, \ldots, X_n$, with $X_{0:n} = 0$, and define

$$D_{i,n} = (n - i + 1)(X_{i:n} - X_{i-1:n}),\ i = 1, 2, \ldots, n, \tag{1.1}$$

$$S_{i,n} = \sum_{j=1}^{i} D_{j,n},\ i = 1, 2, \ldots, n, \tag{1.2}$$

and

$$\underset{\sim}{W}_{r,n} = \left(\frac{S_{1,n}}{S_{r,n}}, \frac{S_{2,n}}{S_{r,n}}, \cdots \frac{S_{r-1,n}}{S_{r,n}}\right). \tag{1.3}$$

If the X_i's are iid (i.e. independent and identically distributed) and $r = n$, then it follows from Seshadri, Csörgő and Stephens (1969) and Dufour, Maag and van Eeden (1984) that $\underset{\sim}{W}_{r,n}$ is distributed as the vector of order statistics relative to a random sample (i.e. a sample with independent observations) of size $r - 1$ from the uniform distribution on $(0, 1)$ only if X_i's are exponential. (Incidentally there is a mistake in the Seshadri etal proof for the result; Dufour et al point out this and supply a correct proof.) Dufour (1982) conjectured that this latter result also holds if $r = n$ is replaced by $2 \leq r < n$ (see, for example, Leslie and van Eeden (1993)).

Leslie and van Eeden (1993) proved that the Dufour conjecture hold if $r \geq \frac{2n}{3} + 1$; note that in the case $2 \leq r < n$, the condition $r \geq \frac{2n}{3} + 1$ implies that $r \geq 5$ and $n \geq 6$. More recently Xu and Yang (1995) and, independently, Rao and Shanbhag (1995) proved that the conjecture is true if $r \geq 5$. Rao and Shanbhag (1995) also established that a certain uniqueness theorem related to the conjecture, extending the Leslie and van Eeden result holds.

The purpose of the present paper is to report the Rao-Shanbhag uniqueness theorem mentioned above, and throw further light on the structural aspects of $\underset{\sim}{W}_{r,n}$ by showing, in particular, that if $r \geq 5$ and X_i's have certain exchangeability property then $\underset{\sim}{W}_{r,n}$ is distributed as the vector of order statistics relative to a random sample of size $r - 1$ from the uniform distribution on $(0, 1)$ only if the distribution of X_i is a mixture of exponential distributions. The latter result appearing here subsumes the result for $r \geq 5$ of Xu and Yang (1995) and Rao and Shanbhag (1995) on the Dufour conjecture, in the case when X_i's are iid.

2. THE RESULTS

Using essentially the techniques used in Kingman (1972) and Kotlarski (1967), we now give our theorems. (For further applications of the techniques, see Rao and Shanbhag (1994) and Prakasa Rao (1992) and the relevant references cited in these monographs.)

Theorem 1: Let r and n be integers such that $5 \leq r \leq n$ and $(\Omega, \mathcal{E}, P)$ be a probability space. Let $X_1, X_2, ..., X_n$ be positive random variables defined on the space $(\Omega, \mathcal{E}, P)$ and $\mathcal{F}$ be a sub-σ-field of $\mathcal{E}$ such that conditional upon $\mathcal{F}$, the random variables $X_1, X_2, ..., X_n$ are iid. Then (in the notation given in section 1) $\underset{\sim}{W}_{r,n}$ is distributed as the vector of order statistics relative to a random sample of size $r-1$ from the uniform distribution on $(0,1)$ if and only if there exists a positive $\mathcal{F}$-mea surable random variable Λ such that conditionally upon it, X_1 is exponentially distributed with mean Λ^{-1}.

Proof: The "if" part follows easily on noting, amongst other things, that (with Λ having stated properties) if, conditionally upon Λ, X_1 is exponential with mean Λ^{-1}, then conditionally upon Λ, X_i's are iid exponential random variables. (Note that if X_i's are as observed here then we have $\underset{\sim}{W}_{r,n}$ to be independent of Λ, with distribution as stated in the assertion.) To prove now the "only if" part of the assertion, assume that $\underset{\sim}{W}_{r,n}$ has the uniform-order-statistics distribution in question. Note that the distribution of $\underset{\sim}{W}_{r,n}$ determines that of $\left(\frac{X_{1:n}}{X_{5:n}}, \frac{X_{2:n}}{X_{5:n}}, \frac{X_{3:n}}{X_{5:n}}, \frac{X_{4:n}}{X_{5:n}}\right)$ (irrespectively of what $(\Omega, \mathcal{E}, P)$, $\mathcal{F}$, $X_1, X_2, ..., X_n$ are provided the stated assumptions are met) and we have here

$$\begin{aligned} 0 &= P\{X_{1:n} = X_{2:n}\} \\ &= E(P\{X_{1:n} = X_{2:n} \mid \mathcal{F}\}). \end{aligned} \tag{2.1}$$

(The notation used is as in section 1.) (2.1) implies that

$$P\{X_{1:n} = X_{2:n} \mid \mathcal{F}\} = 0 \quad a.s.$$

and hence it follows that there is a version of the conditional distribution of X_1 given $\mathcal{F}$ such that it is continuous. Denote this version of the conditional distribution by $F(. \mid \mathcal{F})$. Define now

$$\begin{aligned} \Delta(x_1, x_2, x_3, x_4) &= E\left(\int_{(0,\infty)} (\prod_{j=1}^{4} \overline{F}(yx_j \mid \mathcal{F}))(\overline{F}(y \mid \mathcal{F}))^{n-5} dF(y \mid \mathcal{F})\right), \\ & 0 \leq x_1 \leq x_2 \leq x_3 \leq x_4 \leq 1, \end{aligned} \tag{2.2}$$

where

$$\overline{F}(x \mid \mathcal{F}) = 1 - F(x \mid \mathcal{F}),\ x \in [0,\infty).$$

Since

$$\prod_{j=1}^{4} \overline{F}(yx_j \mid \mathcal{F})) = \prod_{j=1}^{4} \{\overline{F}(y \mid \mathcal{F}) + \sum_{k=j}^{4} (F(yx_{k+1} \mid \mathcal{F}) - F(yx_k \mid \mathcal{F}))\},$$
$$0 \le x_1 \le x_2 \le x_3 \le x_4 \le 1,\ y > 0, \tag{2.3}$$

where $x_5 = 1$, we can express $\Delta(x_1, x_2, x_3, x_4)$ (with x_i's in question as in (2.2)) as a linear combination of quantities

$$E\Big(\int_{(0,\infty)} \Big(\prod_{k=1}^{4} (F(yx_{k+1} \mid \mathcal{F}) - F(yx_k \mid \mathcal{F}))^{l_k}\Big)(\overline{F}(y \mid \mathcal{F}))^{n+l_5-5} dF(y \mid \mathcal{F})\Big)$$

with coeffiecients depending only on $(l_1, l_2, l_3, l_4, l_5)$ where $l_1,\ l_2,\ l_3,\ l_4$ and l_5 are nonnegative integers such that $l_1 \le 1, l_2 \le 2, l_3 \le 3, l_4, l_5 \le 4$ and $l_1 + l_2 + l_3 + l_4 + l_5 = 4$. Consequently, it follows that the function Δ is determined by the distr ibution of $\left(\frac{X_{1:n}}{X_{5:n}}, \frac{X_{2:n}}{X_{5:n}}, \frac{X_{3:n}}{X_{5:n}}, \frac{X_{4:n}}{X_{5:n}}\right)$. This, in turn implies that for $z_1, z_2 \in [0,1]$ with $z_1 + z_2 \le 1$,

$$E\Big(\int_{(0,\infty)} \Big(\overline{F}(yz_1 \mid \mathcal{F})\overline{F}(yz_2 \mid \mathcal{F}) - \overline{F}(y(z_1+z_2) \mid \mathcal{F})\Big)^2 (\overline{F}(y \mid \mathcal{F}))^{n-5} dF(y \mid \mathcal{F})\Big)$$

is determined by the distribution of the vector in question. The expectation appearing immediately above equals zero if $F(. \mid \mathcal{F})$ is identically equal to the distribution function of an exponential distribution. (The existence of the case with $F(. \mid \mathcal{F})$ as mentioned here is obvious.) Consequently, it follows that in the present situation with $\underset{\sim}{W}_{r,n}$ having the stated uniform order-statistics distribution, we should have the expectation to be equal to zero even when $F(. \mid \mathcal{F})$ is not of the form just mentioned. In view of this, we get

$$\overline{F}(yz_1 \mid \mathcal{F})\overline{F}(yz_2 \mid \mathcal{F}) = \overline{F}(y(z_1+z_2) \mid \mathcal{F}),$$
$$0 \le z_1, z_2 \le 1,\ z_1 + z_2 \le 1,\ y \in supp[F(. \mid \mathcal{F})],\ a.s. \tag{2.4}$$

(2.4) implies that we can indeed modify $F(. \mid \mathcal{F})$ appropriately (still retaining its continuously) so that the following condition is met:

$$\overline{F}(yz_1 \mid \mathcal{F})\overline{F}(yz_2 \mid \mathcal{F}) = \overline{F}(y(z_1+z_2) \mid \mathcal{F}),\ 0 \le z_1, z_2 \le 1,\ z_1 + z_2 \le 1,\ y \in supp[F(. \mid \mathcal{F})]. \tag{2.5}$$

(2.5) implies that $(\overline{F}(\frac{y}{2} \mid \mathcal{F}))^2 = \overline{F}(y \mid \mathcal{F})$, $y \in supp[F(. \mid \mathcal{F})]$, and hence that the right extremity of $F(. \mid \mathcal{F})$ equals ∞. Appealing to (2.5), we can then immediately conclude that $\overline{F}(. \mid \mathcal{F})$ satisfies the Cauchy equation on $[0, \infty)$ and hence

$$\overline{F}(y \mid \mathcal{F}) = (\overline{F}(1 \mid \mathcal{F}))^y, \ y \in (0, \infty) \tag{2.6}$$

with $\overline{F}(1 \mid \mathcal{F}) \in (0,1)$. Taking

$$\Lambda = -\ln \overline{F}(1 \mid \mathcal{F}), \tag{2.7}$$

we can then conclude that the result sought holds.

Remark 1: In the case when $X_1, \ X_2, ..., X_n$ are the first n memebers of an infinite sequence $\{X_m\}$ of exchangeable random variables, then the assertion of Theorem 1 holds with $\mathcal{F}$ as the tail or invariant σ-field relative to $\{X_m\}$.

Remark 2: From the proof of Theorem 1, it is clear that the theorem holds if the portion "$\underset{\sim}{W}_{r,n}$ is distributed as ... the uniform distribution on $(0,1)$"is replaced by that " $\left(\frac{X_{1:n}}{X_{r:n}}, , \cdots \frac{X_{4:n}}{X_{r:n}}\right)$ has the distribution as in the case of a random sample of size n from an exponential distribution,"

Theorem 2: Let X_j's be iid, and r and n be such that $n \geq 3$ and $r \geq (2/3)n + 1$. Then, the distribution of $\underset{\sim}{W}_{r,n}$ determines that of X_1 up to a change of scale if there exists a continuous distribution relative to X_j's that gives the distribution of $\underset{\sim}{W}_{r,n}$ as in the present case, such that the characteristic function of $\min\limits_{1 \leq j \leq n-r+1} (\log X_j)$ is nonvanishing or the corresponding distribution has moments of all orders with these determining the distribution.

Proof: Let $X_1, X_2, \ldots, X_n$ be iid positive random variables for which the distribution of $\underset{\sim}{W}_{r,n}$ is as given, and let Y_1, Y_2 and Y_3 be iid random variables distributed as $\min\limits_{1 \leq j \leq n-r+1} X_j$. As there exists a continuous distribution relative to X_j giving the distribution of $\underset{\sim}{W}_{r,n}$ as required, it follows that $P\{\underset{\sim}{W}_{r,n} = (1, 1, \ldots, 1)\} = 0$ (with obviously the vector of 1's having $r-1$ components), it follows that we can assume X_j's to be continuous. Assume then that the distribution of $\underset{\sim}{W}_{r,n}$ is known and X_j's are continuous. Denoting the distribution function of X_1 by F, we see that

$$\begin{aligned} & 2P\left\{\frac{Y_1}{Y_3} \in (0, z_1), \frac{Y_2}{Y_3} \in (0, z_2), Y_1 < Y_2\right\} \\ = (n-r+1)^3 & \int_0^{\infty} \int_0^{y_3 z_2} \int_0^{\min\{y_3 z_1, y_2\}} \Big\{(1-F(y_1))(1-F(y_2))(1-F(y_3))\Big\}^{n-r} \\ & dF(y_1)dF(y_2)dF(y_3), \ z_1, z_2 \in (0,1). \end{aligned} \tag{2.8}$$

Noting that

$$
\begin{aligned}
&\Big\{(1-F(y_1))(1-F(y_2))(1-F(y_3))\Big\}^{n-r}\\
&=\Big\{\Big(1-F(y_3)+F(y_3)-F(y_2)+F(y_2)-F(y_1)\Big)\\
&\quad\Big(1-F(y_3)+F(y_3)-F(y_2)\Big)\Big(1-F(y_3)\Big)\Big\}^{n-r}\\
&\cdot\Big(1-F(y_3)+F(y_3)-F(y_2)+F(y_2)-F(y_1)+F(y_1)\Big)^{3r-2n-3},
\end{aligned}
$$

we can express, in view of the binomial theorem, $\{(1-F(y_1))(1-F(y_2))(1-F(y_3))\}^{n-r}$ as a linear combination of the quantities $(F(y_1))^{\ell_1}(F(y_2)-F(y_1))^{\ell_2}(F(y_3)-F(y_2))^{\ell_3}(1-F(y_3))^{\ell_4}$ with coefficients known and independent of y_1, y_2, y_3, where $\ell_1, \ell_2, \ell_3, \ell_4$ are nonnegative integers such that $\ell_4 \geq n-r$ and $\ell_1+\ell_2+\ell_3+\ell_4 = n-3$. Consequently, in view of (2.8) and Fubini's theorem, we get that for $z_1, z_2 \in (0,1), P\{\frac{Y_1}{Y_3} \in (0,z_1), \frac{Y_2}{Y_3} \in (0,z_2), Y_1 < Y_2\}$ can be expressed as a linear combination of

$$
P\left\{\frac{X_{\ell_1+1:n}}{X_{n-\ell_4:n}} \in (0,z_1), \frac{X_{\ell_1+\ell_2+2:n}}{X_{n-\ell_4:n}} \in (0,z_2)\right\}
$$

with coefficients known (and independent of z_1, z_2), where $\ell_1, \ell_2, \ell_3, \ell_4$ are as mentioned above; this follows on observing, amongst other things, that for the $z_1, z_2, \ell_1, \ell_2, \ell_3, \ell_4$ in question,

$$
\begin{aligned}
&P\left\{\frac{X_{\ell_1+1:n}}{X_{n-\ell_4:n}} \in (0,z_1), \frac{X_{\ell_1+\ell_2+2:n}}{X_{n-\ell_4:n}} \in (0,z_2)\right\}\\
&=\frac{n!}{\ell_1!\ell_2!\ell_3!\ell_4!}\int_0^\infty\int_0^{y_3z_2}\int_0^{\min\{y_3z_1,y_2\}}(F(y_1))^{\ell_1}(F(y_2)-F(y_1))^{\ell_2}(F(y_3)-F(y_2))^{\ell_3}\\
&\qquad(1-F(y_3))^{\ell_4}dF(y_1)dF(y_2)dF(y_3). \qquad (2.9)
\end{aligned}
$$

This, in turn, implies that $P\left\{\frac{Y_1}{Y_3} \in (0,z_1), \frac{Y_2}{Y_3} \in (0,z_2), Y_1 < Y_2\right\}, z_1, z_2 \in (0,1)$ is determined by the distribution of $\underset{\sim}{W}_{r,n}$ (because for each $z_1, z_2, \ell_1, \ell_2, \ell_3, \ell_4$ meeting the stated constraints, the left hand side of (2.9) is determined by the distribution referred to). By symmetry, the result holds when $Y_1 < Y_2$ (under the probability sign) is replaced by $Y_2 < Y_1$,

and, hence when $Y_1 < Y_2$ is deleted. We have

$$
\begin{aligned}
& E\left\{ e^{it_1 \log(Y_1/Y_3) + it_2 \log(Y_2/Y_3)} \right\} \\
=& E\left\{ e^{it_1 \log(Y_1/Y_3) + it_2 \log(Y_2/Y_3)} I_{\{Y_1 < Y_3, Y_2 < Y_3\}} \right\} \\
+& E\left\{ e^{it_1 \log(Y_1/Y_3) + it_2 \log(Y_2/Y_3)} I_{\{Y_2 < Y_1, Y_3 < Y_1\}} \right\} \\
+& E\left\{ e^{it_1 \log(Y_1/Y_3) + it_2 \log(Y_2/Y_3)} I_{\{Y_1 < Y_2, Y_3 < Y_2\}} \right\}, t_1, t_2 \in \mathcal{R}.
\end{aligned} \tag{2.10}
$$

The first term on the right hand side of (2.10) is determined for each $(t_1, t_2) \in \mathcal{R}^2$ by $P\left\{ \frac{Y_1}{Y_3} \in (0, z_1), \frac{Y_2}{Y_3} \in (0, z_2) \right\}, z_1, z_2 \in (0,1)$ and hence by the distribution of $\underset{\sim}{W}_{r,n}$. As

$$
\begin{aligned}
& E\left\{ e^{it_1 \log(Y_1/Y_3) + it_2 \log(Y_2/Y_3)} I_{\{Y_2 < Y_1, Y_3 < Y_1\}} \right\} \\
=& E\left\{ e^{i(-t_1 - t_2) \log(Y_3/Y_1) + it_2 \log(Y_2/Y_1)} I_{\{Y_2 < Y_1, Y_3 < Y_1\}} \right\}, t_1, t_2 \in \mathcal{R}
\end{aligned}
$$

(and Y_j's are iid), we can then see that for each $(t_1, t_2) \in \mathcal{R}^2$, the second term on the right hand side of (2.10) is determined (by the distribution of $\underset{\sim}{W}_{r,n}$); by symmetry, it also follows that this last result holds if we take "the third term" in place of "the second term". Consequently, from (2.10) we have that the characteristic function, or equivalently the distribution, of $(\log(Y_1/Y_3), \log(Y_2/Y_3))$ is determined. If there exists a distribution relative to X_j such that the characteristic function of $\log Y_1$ is nonvanishing, then, in view of the result of Kotlarski (1967) appearing as Theorem 2.1.1 on page 8 in Prakasa Rao (1992), we get that the distribution of $\log Y_1$, but for a change of location, or equivalently that of X_1, but for a change of scale is determined. On the other hand, if there exists a distribution relative to X_j such that the distribution of $\log Y_1$, has moments of all orders with these determining the distribution, then the distribution of $\log\left(\frac{Y_1}{Y_3}\right) + \log\left(\frac{Y_2}{Y_3}\right)$ (i.e. of $\log Y_1 + \log Y_2 - 2\log Y_3$) determines that of $\log Y_1$ up to a change of location, or equivalently of X_1 up to a change of scale. (Note that the distribution of $\log Y_1 + \log Y_2 - 2\log Y_3$ here implies $(\log Y_1)^m$ to be integrable for each nonnegative integer m, and it determines inductively the moment sequence of the distribution of $\log Y_1$ given the corresponding mean.) Hence the proof of the Theorem 2 is complete.

Corollary 1: If X_j's are iid, and $r = n = 3$ or $r = n = 4$ or $r, n \geq 5$, then $\underset{\sim}{W}_{r,n}$ is distributed as the vector of order statistics relative to a random sample of size $r - 1$ from the uniform distribution on (0,1) if and only if X_1 is an exponential random variable.

Proof: In view of Theorem 1, it is sufficient if we prove the result in the case when $r, n \in \{3,4\}$ with $r = n$. In this latter case the "if" part is trivial and the "only if" part follows easily on noting that if Z is exponential, then $\log Z$ has moments of all orders with these determining the distribution. (The "only if" part also follows from the fact that if Z is exponential, then the characteristic function of $\log Z$ is nonvanishing; indeed, the characteristic function of $\log Z$ here is self-decomposable, see, for example, Shanbhag and Sreehari (1977).)

Remark 9: The following example illustrates that Theorem 2 does not hold if the assumption on the distribution $\underset{\sim}{W}_{r,n}$ is dropped.

Example: Let

$$\tau(x) = \begin{cases} 1 - \mid x \mid & \text{if} \quad \mid x \mid \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\Phi(t) = \tau(t) + \frac{e^{i}}{2}\tau(t+3) + \frac{e^{-i}}{2}\tau(t-3), \; t \in \mathcal{R}$$

and

$$\Psi(t) = \tau(t) + \frac{1}{2}\tau(t+3) + \frac{1}{2}\tau(t-3), \; t \in \mathcal{R}$$

Note that the Φ and Ψ defined here are characteristic functions that are not nonvanishing. Let X_1, X_2 and X_3 be positive iid random variable such that the characteristic functions of $\ln X_1$ is Φ, and X_1^*, X_2^* and X_3^* be positive iid random variables such that the characteristic function of $\ln X_1^*$ is Ψ. We have here

$$\Phi(t_1)\Phi(t_2)\Phi(-t_1 - t_2) = \Psi(t_1)\Psi(t_2)\Psi(-t_1 - t_2), \; t_1, t_2 \in \mathcal{R},$$

implying that

$$(\frac{X_1}{X_3}, \; \frac{X_2}{X_3}) \stackrel{d}{=} (\frac{X_1^*}{X_3^*}, \; \frac{X_2^*}{X_3^*});$$

we have also here the distributions of X_i and X_i^* to be continuous (indeed absolutely continuous with respect to Lebesgue measure). It now follows easily that the distribution of $W_{3,3}$ with X_i's as in the example, equals, that of its analogue with X_i^*'s in place of X_i's. Consequently, we have that Theorem 2 does not hold if the portion ", such that ... these determining the distribution "in its statement is deleted.

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